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BANACH AND ABSOLUTE BANACH LIMITS OF DOUBLE SEQUENCES

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Abstract

The idea of Banach limit for double sequences has been recently introduced in [M. Mursaleen and S. A. Mohiuddine, Banach limit and some new spaces of double sequences, Turkish J. Math. 36(1) (2012), 121-130]. In this paper, we determine the sublinear functionals to associate with the concepts of strong almost convergence and absolute almost convergence for double sequences. We characterize the space of absolute almost convergent double sequences through a new notion, that is, absolute Banach limit.

1. Introduction and Preliminaries

Recently the concept of Banach limits for double sequences has been introduced by Mursaleen and Mohouddine [12] as follows.

Definition 1.1. A linear functional $\mathcal{L}$ on $\mathcal{M}_u$ is said to be a *Banach limit* if it has the following properties:

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(i) $\mathcal{L}(x) \geq 0$ if $x \geq 0$ (i.e., $x_{jk} \geq 0$, for all j, k),

(ii) $\mathcal{L}(\mathbf{e}) = 1$, where $\mathbf{e} = (e_{jk})$ with $e_{jk} = 1$, for all j, k, and

(iii) $\mathcal{L}(S_{11}x) = \mathcal{L}(x) = \mathcal{L}(S_{10}x) = \mathcal{L}(S_{01}x)$,

where the shift operators S_{01}, S_{10} and S_{11} are defined by

$$S_{01}x = (x_{j,k+1}), \; S_{10}x = (x_{j+1,k}), \; S_{11}x = (x_{j+1,k+1}).$$

Let $\mathcal{B}_2$ be the set of all Banach limits on $\mathcal{M}_u$. A double sequence $x = (x_{jk})$ is said to be *almost convergent* to a number ℓ if $\mathcal{L}(x) = \ell$, for all $\mathcal{L} \in \mathcal{B}_2$; where $\mathcal{M}_u$ denotes the space of all bounded double sequences $x = (x_{jk})$ of real or complex numbers, i.e.,

$$\mathcal{M}_u = \{x = (x_{jk}) : \| x \|_\infty = \sup_{j,k} | x_{jk} | < \infty\}.$$

More precisely, $\mathcal{M}_u$ is the space of all uniformly bounded double sequences.

The idea of almost convergence for double sequences was introduced and studied by Moricz and Rhoades [8].

Definition 1.2. A double sequence $x = (x_{jk})$ is said to be *almost convergent* to a limit ℓ if

$$\lim_{r,q \to \infty} \sup_{s,t > 0} \rho_{rqst}(x) = \ell, \tag{$*$}$$

where $\rho_{rqst}(x) = \frac{1}{rq} \sum_{j=s}^{s+r-1} \sum_{k=t}^{t+q-1} x_{jk}$. In this case ℓ is called the $\mathcal{F}$-limit of x and we shall denote by $\mathcal{F}$ the space of all almost convergent double sequences, that is,

$$\mathcal{F} = \{x = (x_{jk}) \in \mathcal{M}_u : \lim_{p,q \to \infty} \rho_{pqst}(x) = \ell \text{ uniformly in } s, t; \; \ell = \mathcal{F}\text{-}\lim x\}. \tag{1.1}$$

As in case of single sequences [6], equivalence of these two definitions can be proved similarly.

If $s = t = 1$ in $(*)$, then we get $(C, 1, 1)$-convergence, and in this case we write $x_{jk} \to \ell(C, 1, 1)$, where $\ell = (C, 1, 1)\text{-}\lim x$.

A double sequence $x = (x_{jk})$ is said to converge to the limit L in Pringsheim's sense (shortly, p-convergent to L) if for every $\varepsilon > 0$ there exists an integer N such that $|x_{jk} - L| < \varepsilon$ whenever $j, k > N$. In this case L is called the p-limit of x. If in addition $x \in \mathcal{M}_u$, then x is said to be boundedly convergent to L in Pringsheim's sense (shortly, bp-convergent to L).

A double sequence $x = (x_{jk})$ is said to converge regularly to L (shortly, r-convergent to L) if x is p-convergent and the limits $x_j := \lim_k x_{jk} (j \in \mathbb{N})$ and $x^k := \lim_j x_{jk} (k \in \mathbb{N})$ exist. Note that in this case, the limits $\lim_j \lim_k x_{jk}$ and $\lim_k \lim_j x_{jk}$ exist and are equal to the p-limit of x.

In general, for any notion of convergence ν, the space of all ν-convergent double sequences will be denoted by $\mathcal{C}_\nu$ and the limit of a ν-convergent double sequence x by $\nu\text{-}\lim_{j,k} x_{jk}$, where $\nu \in \{p, bp, r\}$.

Note that in contrast to the single sequences, a p-convergent double sequence need not be almost convergent. However, every bounded convergent double sequence is almost convergent and every almost convergent double sequence is also bounded, i.e. $\mathcal{C}_{bp} \subset \mathcal{F} \subset \mathcal{M}_u$ and each inclusion is proper. Also $\mathcal{F}$ is a Banach space with $\|\cdot\|_\infty$ since $\mathcal{C}_{bp}$ and $\mathcal{M}_u$ are Banach spaces with $\|\cdot\|_\infty$.

In this paper, we determine the sublinear functionals to associate with the concepts of strong almost convergence and absolute almost convergence for double sequences. We characterize the space of absolute almost convergent double sequences through a new notion, that is, absolute Banach limit.

2. Strong Almost Convergence

The idea of strong almost convergence for double sequences was given by Basarir [1]. In this section, we define the concepts of absolute almost convergence for double sequences. These concepts for single sequences were studied in [2] and [3].

Remark 2.1 [11]**.** The sublinear functional $V : \mathcal{M}_u \to \mathbb{R}$ defined on $\mathcal{M}_u$ dominates and generates the Banach limit, where

$$V(x) = \inf_{z=(z_{jk})\in\mathcal{F}_0} \limsup_{j,k} (x_{jk} + z_{jk}).$$

The space $[\mathcal{F}]$ of strongly almost convergent double sequences was defined in [1] as

$$[\mathcal{F}] = \left\{ x = (x_{jk}) \in \mathcal{M}_u : \lim_{p,q\to\infty} \frac{1}{(p+1)(q+1)} \sum_{j=0}^{p}\sum_{k=0}^{q} | x_{j+s,k+t} - \ell | = 0, \text{ uniformly in } s, t \right\}.$$

Here we investigate the sublinear functional which generates the space $[\mathcal{F}]$.

Definition 2.2. Consider the sublinear functional $\Psi : \mathcal{M}_u \to \mathbb{R}$ defined by

$$\Psi(x) = \limsup_{r,q} \sup_{s,t} \frac{1}{(r+1)(q+1)} \sum_{j=0}^{r}\sum_{k=0}^{q} | x_{j+s,k+t} |.$$

Let us write $\{\mathcal{M}_u, \Psi\}$ for the set of all linear functionals Φ on $\mathcal{M}_u$ such that $\Phi(x) \le \Psi(x)$ for all $x = (x_{jk}) \in \mathcal{M}_u$. By Hahn-Banach Theorem, the set $\{\mathcal{M}_u, \Psi\}$ is non-empty.

If there exists $\ell \in \mathbb{R}$ such that

$$\Phi(x - \ell\mathbf{e}) = 0 \text{ for all } \Phi \in \{\mathcal{M}_u, \Psi\} \tag{2.2}$$

then we say that x is $\{\mathcal{M}_u, \Psi\}$-convergent to ℓ and in this case we write $\{\mathcal{M}_u, \Psi\}$-$\lim x = \ell$.

Now we prove the following result:

Theorem 2.3. $[\mathcal{F}]$ *is the set of all* $\{\mathcal{M}_u, \Psi\}$*-convergent sequences.*

Proof. Let $x \in [\mathcal{F}]$. Then for each $\varepsilon > 0$, there exist p_0, q_0 such that for $r > r_0$, $q > q_0$ and all s, t,

$$\frac{1}{(r+1)(q+1)} \sum_{j=0}^{r}\sum_{k=0}^{q} | x_{j+s,k+t} - \ell | < \varepsilon,$$

and this implies that $\Psi(x - \ell\mathbf{e}) \le \varepsilon$. In a similar manner, we can prove that $\psi(\ell\mathbf{e} - x) \le \varepsilon$. Hence $| \Phi(x - \ell\mathbf{e}) | \le \Psi(x - \ell\mathbf{e}) \le \varepsilon$, for all $\Phi \in \{\mathcal{M}_u, \Psi\}$.

Therefore, $\Phi(x - \ell\mathbf{e}) = 0$, for all $\Phi \in \{\mathcal{M}_u, \Psi\}$ and this implies that by (2.2) $x \in [\mathcal{F}]$ implies that x is $\{\mathcal{M}_u, \Psi\}$- convergent.

Conversely, suppose that x is $\{\mathcal{M}_u, \Psi\}$-convergent, that is,

$$\Phi(x - \ell\mathbf{e}) = 0, \text{ for all } \Phi \in \{\mathcal{M}_u, \Psi\}.$$

Since Ψ is sublinear functional on $\mathcal{M}_u$, by Hahn-Banach Theorem, there exists $\Phi_0 \in \{\mathcal{M}_u, \Psi\}$ such that $\Phi_0(x - \ell\mathbf{e}) = \Psi(x - \ell\mathbf{e})$. Hence $\Psi(x - \ell\mathbf{e}) = 0$; and since $\Psi(x) = \Psi(-x)$, it follows that $x \in [\mathcal{F}]$.

This completes the proof of the theorem.

3. Absolute Almost Convergence and Absolute Banach Limit

Write

$$\phi_{rqst}(x) = \rho_{rqst}(x) - \rho_{r-1\,q,s,t}(x) - \rho_{r,q-1,s,t}(x) + \rho_{r-1,q-1,s,t}(x).$$

Thus simplifying further, we get

$$\phi_{rqst}(x)$$

$$= \frac{1}{r(r+1)\,q(q+1)} \sum_{m=1}^{r} \sum_{n=1}^{q} mn[x_{m+s,n+t} - x_{m+s-1,n+t} - x_{m+s,n+t-1} + x_{m+s-1,n+t-1}].$$

Now we write

$$\phi_{rqst}(x)$$

$$= \begin{cases} \frac{1}{r(r+1)\,q(q+1)} \sum_{m=1}^{r} \sum_{n=1}^{q} mn[x_{m+s,n+t} \\ - x_{m+s-1,n+t} - x_{m+s,n+t-1} + x_{m+s-1,n+t-1}]; & r, q \geq 1 \\ x_{st}; & r = 0 \text{ or } q = 0 \text{ or both } r, q = 0. \end{cases} \tag{3.1}$$

Recently, the idea of almost bounded variation was defined in [10] as follows:

Definition 3.2. A double sequence $x = (x_{jk}) \in \mathcal{M}_u$ is said to be *almost bounded variation* if and only if

(i) $\sum_{r=0}^{\infty}\sum_{q=0}^{\infty} | \phi_{rqst}(x) |$ converges uniformly in s, t, and

(ii) $bp - \lim_{p,q\to\infty} \rho_{rqst}(x)$, which must exist, should take the same value for all s, t. We denote by $\hat{\mathcal{BV}}$, the space of all absolutely almost convergent double sequences.

By relaxing the condition (ii) in the above definition, we define the following:

Definition 3.2. A double sequence $x = (x_{jk}) \in \mathcal{M}_u$ is said to be of *absolutely almost convergent* if

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} | \phi_{pqst}(x) | \text{ converges uniformly in } s, t.$$

By $\hat{\mathcal{L}}$, we denote the space of all double sequences which are of almost bounded variation.

It is obvious that $\hat{\mathcal{BV}} \subset \hat{\mathcal{L}} \subset [\mathcal{F}] \subset \mathcal{F} \subset \mathcal{M}_u$.

Theorem 3.3 [10]**.** *$\hat{\mathcal{L}}$ and $\hat{\mathcal{BV}}$ both are Banach spaces normed by*

$$\| x \| = \sup_{s,t} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} | \phi_{pqst}(x) |.$$

Remark 3.4. It may be remarked that we have a class of linear continuous functionals Θ on $\mathcal{M}_u$ (which we call the set of Banach limits) such that Θ is uniquely determined if and only if $x \in \mathcal{F}$. Now we are going to deal with the similar situation prevails for $\hat{\mathcal{L}}$.

Theorem 3.5. *There does not exist a class of continuous linear functionals Θ on $\mathcal{M}_u$ such that Θ is uniquely determined if and only if $x \in \hat{\mathcal{L}}$.*

Proof. We first note that $\hat{\mathcal{L}}$ is not closed in $\mathcal{M}_u$. Given the value of $\varphi_2(x)$ for $x \in \hat{\mathcal{L}}$, its value for $x \in cl(\hat{\mathcal{L}})$ are determined by continuity. So if $\varphi_2(x)$ is unique for $x \in \hat{\mathcal{L}}$, it must be unique in the set $cl(\hat{\mathcal{L}})$, which is larger than $\hat{\mathcal{L}}$.

Remark 3.6. As in Remark 2.1, it is easy to see that the sublinear functional

$$\lambda(x) = \lim_{p,q} \sup_{s,t} \sup \tau_{pqst}(x) \tag{3.2}$$

both dominates and generates the σ-means φ_2 is a σ-mean if and only if

$$-\lambda(-x) \leq \varphi_2(x) \leq \lambda(x). \tag{3.3}$$

It follows from (3.3) that φ_2 is unique σ-mean if and only if

$$\mathcal{F} = \{x \in \mathcal{M}_u : \lambda(x) = -\lambda(-x)\}. \tag{3.4}$$

In the same vein, we seek a characterization of a class of linear functionals ψ_2 on $\mathcal{M}_u$ (we can call it absolute Banach limit or $\mathcal{ABL}$) in terms of a suitable sublinear functional Q on $\mathcal{M}_u$; that is, $\psi_2 \in \mathcal{ABL}$ if and only if $-Q(-x) \leq \psi_2(x) \leq Q(x)$ and unique invariant mean if and only if

$$\hat{\mathcal{L}} = \{x \in \mathcal{M}_u : Q(x) = -Q(-x)\}.$$

A suitable candidate for Q appears to be:

$$Q(x) = \lim_{p,q} \sup_{s,t} \sup \sum_{i=p}^{\infty} \sum_{j=q}^{\infty} |\, \phi_{ijst}(x) \,| < \infty. \tag{3.5}$$

Since Q is a sublinear functional on $\mathcal{M}_u$, it follows from Hahn-BanachTheorem that there exists a continuous linear functional μ on $\mathcal{M}_u$ such that

$$\mu(x) \leq Q(x), \quad \text{for all } x \in \mathcal{M}_u \tag{3.6}$$

and this limit is unique if and only if $Q(x) = -Q(-x) = -Q(x)$, i.e., if and only if $Q(x) = 0$, for all $x \in \mathcal{M}_u$, i.e., if and only if

$$\lim_{p,q} \sum_{i=p}^{\infty} \sum_{j=q}^{\infty} |\, \phi_{ijst}(x) \,| = 0, \tag{3.7}$$

i.e., if and only if $x \in \hat{\mathcal{L}}$. Thus we have

Theorem 3.7.

$$\hat{\mathcal{L}} = \{x \in \mathcal{M}_u : Q(x) = 0\}.$$

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