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COSMIC EXPANSION OF A CLOUD OF SPHERICALLY SYMMETRIC PERFECT FLUID TO ULTIMATE REST AT TIME INFINITY

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Abstract

A spherically symmetric solution of Einstein's equations is generated which describes expansion of a cloud of perfect fluid to ultimate rest. The fluid has equation of state $p = \rho$ representing a stiff fluid. For the time range $0 \leq t < \infty$ these physical quantities are everywhere finite, although in the solution there is no imposed upper limit on the density at the centre of symmetry, $r = 0$. Density and pressure tend monotonically to zero as the radial coordinate $r \to \infty$. The fluid comes to rest as $t \to \infty$. The physical, kinematic and geometrical properties of the metric are calculated. The solution is a potential source of gravitational waves and also adds credence to the notion of a multiverse world.

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0. Introduction

Non-static spherically symmetric solutions to the equations of general relativity for a perfect fluid have been obtained by numerous authors. They include McVittie [1, 2], Kustaanheimo and Quist [3], Wyman [4], Srivastava [5, 6], Stephani and Wolf [7], Davidson [8], Lake [9], Van den Bergh and Wils [10]. Most solutions have been without shear, but see [8-10] and the present solution. Here we present a solution in which the fluid is expanding with acceleration and shear.

1. The Metric

Our metric takes the form

$$ds^2 = A(r)(dr^2 - dt^2) + B(r, t)(d\theta^2 + \sin(\theta)^2 d\phi^2), \tag{1.1}$$

where we have chosen coordinates r, *theta*, *phi*, t as x^i for $i = 1, 2, 3, 4,$ respectively. It will be assumed that the metric represents a spherically symmetric perfect fluid of density ρ and pressure p, and that the coordinate system is comoving with it. The 4-velocity of the fluid can therefore be written

$$\nu^a = [0, 0, 0, A(r)^{-1/2}]. \tag{1.2}$$

The energy-momentum tensor for a perfect fluid will be

$$T^{ab} = (\rho + p)v^a v^b + g^{ab} p, \tag{1.3}$$

or in mixed tensor form

$$\begin{bmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & -\rho \end{bmatrix}. \tag{1.4}$$

We now turn to the components of the Einstein tensor, in particular, the component

$$G_1^4 = \frac{\left[\begin{array}{r} 2A(r)B(r,\,t)\dfrac{\partial^2}{\partial r\partial t}\,B(r,\,t) - A(r)\dfrac{\partial}{\partial r}\,B(r,\,t)\dfrac{\partial}{\partial t}\,B(r,\,t) \\ -\,B(r,\,t)\dfrac{d}{dr}\,A(r)\dfrac{\partial}{\partial t}\,B(r,\,t) \end{array}\right]}{2A(r)^2\,B(r,\,t)^2}. \tag{1.5}$$

Referring to (1.4) this component must equate to zero. Evidently also $G_1^1 - G_2^2$ must be zero. On reduction the latter gives

$$G_1^1 - G_2^2$$

$$= \frac{\left(\begin{array}{l} -\,2B(r,\,t)A(r)^3 - B(r,\,t)A(r)^2\,\dfrac{\partial^2}{\partial r^2}\,B(r,\,t) - B(r,\,t)A(r)^2\,\dfrac{\partial^2}{\partial t^2}\,B(r,\,t) \\ +\,A(r)^2\left(\dfrac{\partial}{\partial r}\,B(r,\,t)\right)^2 - A(r)B(r,\,t)^2\,\dfrac{d}{dr^2}\,A(r) \\ +\,B(r,\,t)A(r)\dfrac{d}{dr}\,A(r)\dfrac{\partial}{\partial r}\,B(r,\,t) + B(r,\,t)^2\left(\dfrac{d}{dr}\,A(r)\right)^2 \end{array}\right)}{2B(r,\,t)^2\,A(r)^3}. \tag{1.6}$$

We tentatively set $A(r) = Ke^{ar}$ and then find that we can have both G_1^4 and $G_1^1 - G_2^2$ zero, if we set

$$A(r) = Ke^{ar}, \quad B(r,\,t) = 2Ke^{ar}(M + e^{at})\big/Ma^2, \tag{1.7}$$

where K, M and a are all > 0.

The Einstein tensor can now be written

$$\begin{bmatrix} \dfrac{a^2M^2e^{-ar}}{4K(M+e^{at})^2} & 0 & 0 & 0 \\ 0 & \dfrac{a^2M^2e^{-ar}}{4K(M+e^{at})^2} & 0 & 0 \\ 0 & 0 & \dfrac{a^2M^2e^{-ar}}{4K(M+e^{at})^2} & 0 \\ 0 & 0 & 0 & -\dfrac{a^2M^2e^{-ar}}{4K(M+e^{at})^2} \end{bmatrix}. \tag{1.8}$$

We see that the pressure of the fluid equals the density, so that, we have a stiff fluid. For the time range $0 \leq t < \infty$, p and ρ are everywhere finite, but there is no upper limit to the density at $r = 0$. Then as expansion proceeds p and ρ steadily decrease as either r or t increase and the fluid comes to rest in a vacuous state as $r \to \infty$, and throughout as $t \to +\infty$.

2. Properties of the Metric and Fluid

The fluid is subject to expansion, acceleration and shear:

$$\Theta = \frac{ae^{at}}{K^{1/2}(M + e^{at})e^{ar/2}}. \tag{2.1}$$

$$a^i = \left[\frac{ae^{-ar}}{2K}, 0, 0, 0\right]. \tag{2.2}$$

$$\sigma_{ab} = \begin{bmatrix} \dfrac{-aKe^{a(t+r/2)}}{3K^{1/2}(M + e^{at})} & 0 & 0 & 0 \\ 0 & \dfrac{K^{1/2}e^{a(t+r/2)}}{3aM} & 0 & 0 \\ 0 & 0 & \dfrac{K^{1/2}e^{a(t+r/2)}\sin^2\theta}{3aM} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.3}$$

There are very few spherically symmetric solutions exhibiting shear, and especially few with expansion, acceleration and shear as in the present case, but see Lake [9] and Van den Bergh and Wils [10].

The metric admits a group G_3 on S_2, with Killing vectors

$$\begin{aligned} \xi_1 &= [0, 0, 1, 0], \\ \xi_2 &= [0, -\cos\phi, \sin\phi\cot\theta, 0], \\ \xi_3 &= [0, \sin\phi, \cos\phi\cot\theta, 0]. \end{aligned} \tag{2.4}$$

The θ, ϕ 2-space has constant Gaussian curvature C equal to $1/B(r, t)$. Calculation shows that of the Weyl scalars only ψ_2 is non-zero, so that the Petrov type is D.

Concluding Remarks

From a particular form of spherically symmetric metric, we have generated a solution to the equations of general relativity which describe the expansion of a cloud of perfect fluid of finite density and pressure to ultimate rest.

Physical, kinematic and geometrical properties of the fluid have been calculated and demonstrated.

We emphasise that the fluid expands from a unique centre at $r = 0$. This is in contrast to the big-bang which geometrically has a centre everywhere in accordance with the assumed Cosmological Principle by which the view on the large scale is the same from every galaxy at a given cosmic time. If fluid expansions, such as we model here, of an astrophysical origin, took place randomly from various centre points in the universe then the notion of a multiverse world might be realised.

Also, the stiff fluid described here has the sound speed equal to the velocity of light so that any physical disturbance of the fluid would generate a gravitational wave.

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