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THE QUASI-SYMMETRIC MAPS

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Abstract

The purpose of this article is to study the quasi-conformal maps and to be able to demonstrate lemme1 in order to prepare proof of a theorem in a next article.

1. Introduction

Suppose that G is a bilipschitz (i.e., quasi-isometric) map of n dimensional hyperbolic space $\mathbb{H} = \mathbb{H}^n$ onto itself. We may get such a map, for example, when $n = 2$, by starting with a bilipschitz map of compact surfaces $F : \Sigma_1 \to \Sigma_2$ of genus ≥ 2 equipped with curvature -1 metrics. Then, since $\mathbb{H}$ is the universal cover of both Σ_1 and Σ_2, the surface map F lifts to a bilipschitz map of $\mathbb{H}$ to itself.

Using the Poincaré disk model

$$\left(\mathbb{B}^n,\ (1 - |x|^2)^{-2} dx^2\right),$$

we see that a bilipschitz G will extend continuously to the (*ideal*) boundary sphere

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$\mathbb{S} = \mathbb{S}^{n-1} = \partial\mathbb{H}$ at ∞. However, here the boundary map $g = G|\mathbb{S}$ will not necessarily be Lipschitz. In fact, it almost never is for lifts of surface maps. However, it is *quasi-conformal* for $n \geq 3$ and *quasi-symmetric* for $n = 2$. A homeomorphism $g : \mathbb{S} \to \mathbb{S}$ is quasi-conformal (quasi-symmetric) if

$$\Lambda_g \equiv \sup_{x \in \mathbb{S}} \lim_{r \downarrow 0} \sup \frac{\sup_{y \in \mathbb{S} \cap \partial\mathbb{B}_r(x)} |g(x) - g(y)|}{\inf_{z \in \mathbb{S} \cap \partial\mathbb{B}_r(x)} |g(x) - g(y)|} < \infty.$$

In the quasi-symmetric case with $\mathbb{S} = \mathbb{S}^1 \subset \mathbb{C}$, this simply says that the distance ratios

$$\frac{\left| g(z) - g(ze^{i\theta}) \right|}{\left| g(ze^{-i\theta}) - g(z) \right|}$$

are bounded above and below independent of $z \in \mathbb{S}^1$ and $\theta \in \mathbb{R}$. Note that bilipschitz maps are automatically quasi-conformal (quasi-symmetric). However, for regularity, quasi-conformal (quasi-symmetric) maps are generally only Hölder continuous to some power less than 1 depending on Λ_g.

Conversely to the above discussion, Beurling and Ahlfors [4], Douady and Earle [6], and Tam and Wan [12] proved that any quasi-conformal (quasi-symmetric) map of $\mathbb{S}$ admits a continuous extension to $\mathbb{H}$ that is bilipschitz on $\mathbb{H}$. These results suggested the following question by Royden and others.

Does any quasi-conformal (quasi-symmetric) map *g* of $\mathbb{S}$ admit a harmonic map extension to $\mathbb{H}$?

While the general problem is still open, harmonic extensions were first constructed by P. Li and L.-F. Tam [10], [11] under some assumptions on smoothness of *g* and a pointwise lower bound on the ∇_g (see also Akutagawa [1]). Some non-uniqueness examples were found by Wolf [14] and Li and Tam [10], [11]. Harmonic self-maps of $\mathbb{H}^2$ were studied via their Hopf differentials by Tukia and Väisälä [13]. A few years ago, we worked out the following result [8] (see also different, independent recent proofs by Deane Yang [15]).

Theorem 1. *The set of all quasi-conformal (quasi-symmetric) maps of* $\mathbb{S}$ *admitting an extension to a bilipschitz harmonic map of* $\mathbb{H}$ *is open.*

(Here, we say that a sequence $g_i \to g$ if $\Lambda_{g_i \circ g^{-1}} \to 1$).

Corollary 1. *All quasi-conformal (quasi-symmetric) maps of* $\mathbb{S}$ *near the identity are harmonically extendible.*

For the case $n = 2$, this Corollary was obtained several years ago by C. Earle and S. Fowler [7] using implicit function theorem methods. Proof of Theorem. (For details, see [8]) We start with a bilipschitz harmonic map $\mathbb{H}_0 : \mathbb{H} \to \mathbb{H}$ and consider small quasi-conformal (quasi-symmetric) perturbations of the boundary map $h_0 \equiv \mathbb{H}_0 | \mathbb{S}$. Specifically, we will consider, for $\delta > 0$, a map $g = g_s$ with $\Lambda_{g \circ h_0^{-1}} \leq \delta$. Using one of the constructions of [2], [4], [6] or [12], we extend $g \circ h_0^{-1}$ to a map $F = F_\delta$ of $\mathbb{H}$ with

$$\sup_{\mathbb{H}} d(Id, F) + \sup_{\mathbb{H}} \| Id - dF \| \leq \varepsilon = \varepsilon(\delta), \tag{1}$$

where $d(.,.) = dist_{\mathbb{H}}(.,.)$ and here, and in the following, $\varepsilon(\delta)$ will denote some (changing) positive function of δ which approaches 0 as $\delta | 0$. Then,

$$G = G_\delta \equiv F \circ H_0$$

is a bilipschitz extension of g that is bilipschitz close (as in (1)) to H_0. Our goal is to obtain, for small δ a bilipschitz extension H of g which is also harmonic, that is, has tension

$$\tau(H) = 0.$$

We first observe that, in addition to (1), we may also assume that

$$F \text{ is } C^2 \text{ and } \tau(F) \leq \varepsilon = \varepsilon(\delta). \tag{2}$$

To see this, we may, for example, divide $\mathbb{H}$ into compact isometric n-dimensional blocks, as in a standard dyadic decomposition of the upper half space model with totally geodesic faces. For any one such block B we may, by (1), associate a hyperbolic isometric F_B so that, for all $b \in B$,

$$d(F(b),\ F_B(b)) + \| (dF)_b - (dF_B)_b \| \le \varepsilon = \varepsilon(\delta).$$

On a fixed-size η tubular neighborhood of $n-1$ skeleton we may locally smoothly interpolate between the isometries associated with the blocks of the adjacent faces. One may do this by inductively crossing the $n-1$, then $n-2$, ..., 0 cells. One eventually gets the smooth map $\tilde{F} : \mathbb{H} \to \mathbb{H}$ satisfying $\| \tau(\tilde{F}) \| < C\varepsilon / \eta^n$ and replace F by $\tilde{F}$ to get (2).

Since H_0 is harmonic and bilipschitz, it now follows from (1) and (2) that

$$\tau(G) = \tau(F \circ H_0) \le \varepsilon = \varepsilon(\delta) \tag{3}$$

and

$$\min_{|v|=1} | dG(v) | \ge (1 - \varepsilon(\delta))\mu(H_0), \tag{4}$$

where $\mu(H_0) = \min_{|v|=1} | dH_0(v) |.$

To find the desired harmonic *H*, we consider integer radius balls $\mathbb{B}_1, \mathbb{B}_2, \ldots$ about some fixed point in $\mathbb{H}$ and use [8] to choose, for each $m = 1, 2, \ldots,$ a harmonic map

$$H_m : \mathbb{B}_m \to \mathbb{H} \quad \text{with} \quad H_m = G \ \text{ on } \ \partial\mathbb{B}_m.$$

We want to show that for δ sufficiently small, H_m converges as $m \to \infty$ to the desired *H*. We use the following:

Lemma 1. *If S and T are two nowhere-coinciding C^2 maps from a region $\Omega \subset \mathbb{H}$ to $\mathbb{H}$, then the function*

$$Q \equiv \cosh d(S(.), T(.)) - 1$$

satisfies

$$\Delta Q \ge Q\left(\left[\min_{|v|=1} dS(v)\right]^2 + \left[\min_{|v|=1} dT(v)\right]^2\right) - (| \tau(S) | + | \tau(T) |) \sinh d(S, T). \tag{5}$$

The proof is a calculation which we will sketch later. For now, we use the Lemma to complete the proof of the Theorem.

Defining $Q_m = \cosh d(G, H_m) - 1$, we deduce from (3), (4), (5), and the harmonicity of H_m that

$$\Delta Q_m \geq Q_m(\mu(H_0)(1-\varepsilon)+0) - (\varepsilon + 0)\tanh d(G, H_m)(Q_m - 1)$$

on $\mathbb{B}_m$. Since Q_m vanishes on $\partial\mathbb{B}_m$, there a maximum point $a \in \mathbb{B}_m$ for Q_m. Unless $Q_m \equiv 0$, we have there that

$$0 \geq \Delta Q_m(a) \geq (\mu(H_0)(1-\varepsilon)-\varepsilon)Q_m(a) - \varepsilon,$$

hence

$$\sup_{\mathbb{B}_m} Q_m = Q_m(a) \leq \frac{\varepsilon}{\mu(H_0)(1-\varepsilon)-\varepsilon} < \infty,$$

independent of *m*. In any case, since *G* is Lipschitz, the diameter of the image $H_m(B)$ of any unit ball in *B* in $\mathbb{B}_m$ is uniformly bounded, independent of *m*. The gradient estimate of Cheng [5] and Baird and Kamissoko [3] Lemma 2.1 then gives the bound

$$\sup_{\mathbb{B}_{m-1}} |\nabla H_m| \leq C = C(H_0, \delta) < \infty,$$

independent of *m*. The Ascoli-Arzela theorem allows us to find a subsequence of H_m converging uniformly on compacts to a harmonic map $H : \mathbb{H} \to \mathbb{H}$ which is still at bounded distance from *G*. It follows that *H* has the same asymptotic boundary values as *G*, that is, $H | \mathbb{S} = G | \mathbb{S} = g$, which completes the proof of the theorem. □

Sketch of proof of Lemma. First, we compute for a $\mathbb{C}^2$ map $w : M \to N$ of Riemannian manifolds and a smooth function $f : N \to \mathbb{R}$, the pointwise formula

$$\Delta(f \circ w) = tr_{\{w*e_\alpha\}} Hess f + \langle \nabla f, \tau(w) \rangle_N, \tag{6}$$

where the trace of the Riemannian Hessiaan is taken with respect to the push-forward $\{w * e_\alpha\}$ of an orthonormal frame $\{e_\alpha\}$.

For each point $x \in \Omega$, we choose an orthonormal basis $\{\sigma_1, ..., \sigma_n\}$ of $Tan_{S(x)}$ so that σ_1 is the initial velocity of the geodesic γ going from $S(x)$ to $T(x)$. Then we parallel translate along γ to get the basis $\{\tau_1, ..., \tau_n\}$ of $Tan_{T(x)}$. With respect to the basis $(\sigma_1, 0), ..., (\sigma_n, 0), (0, \tau_1), ..., (0, \tau_n)$ of $Tan_{S(x)} \times Tan_{S(x)}$, $(Hess \cosh d)_{(S(x), T(x))}$ is represented by the matrix

$$A \equiv (\cosh d) Id_{2n\times 2n} + \tilde{A},$$

where $\tilde{A}$ has only nonzero entries $-\cosh d$ at $((\sigma_1, 0), (0, \tau_1))$ and $((0, \tau_1), (\sigma_1, 0))$ and -1 at $((\sigma_i, 0), (0, \tau_i))$ and $((0, \tau_i), (\sigma_i, 0))$ for $i = 2, ..., n$. So, by (6) with $w = (S, T)$ and $f = \cosh d$,

$$\Delta Q = \sum_\alpha \Big\langle A \sum_i [\langle dS(e_\alpha), \sigma_i\rangle\sigma_i + \langle dT(e_\alpha), \tau_i\rangle\tau_i],$$

$$\sum_i [\langle dS(e_\alpha), \sigma_i\rangle\sigma_i + \langle dT(e_\alpha), \tau_i\rangle\tau_i]\Big\rangle + \big\langle \nabla Q, (\tau(S), \tau(T))_{(S, T)}\big\rangle$$

$$= (\cosh d)(\langle \nabla S, \sigma_1\rangle - \langle \nabla T, \tau_1\rangle)^2$$

$$+ (\cosh d - 1)\sum_{i=2}^{n} (\langle \nabla S, \sigma_i\rangle^2 + \langle \nabla T, \tau_i\rangle^2) + \big\langle \nabla Q, (\tau(S), \tau(T))_{(S, T)}\big\rangle$$

$$\geq 0 + Q\sum_{i=2}^{n} (\langle \nabla S, \sigma_i\rangle^2 + 0) - \sinh d(S, T)(|\tau(S)| + |\tau(T)|),$$

where $\langle \nabla S, \sigma_1\rangle$ refers to the component of $\sum_a dS(e_\alpha)$ in the direction σ_1, etc. The last inequality clearly implies inequality (5). □

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