NAMBU-JONA-LASINIO MODEL AND DECONSTRUCTED DIMENSION

NAHOMI KAN, KOICHIRO KOBAYASHI[!] and KIYOSHI SHIRAISHI[*!]

Yamaguchi Junior College
Hofu-Shi, Yamaguchi 747-1232
Japan

[!]Yamaguchi University
Yamaguchi-Shi
Yamaguchi 753-8512
Japan

Abstract

The Nambu-Jona-Lasinio model with the mass matrix which appears in a deconstruction model is investigated. We consider two models. In Model A, a mass matrix belonging to a type used in dimensional deconstruction is introduced. In Model B, the four-fermion interaction has a structure of the matrix of the type of dimensional deconstruction. In these models, we find that generation of a dynamical fermion mass spectrum occurs in a strong coupling case.

1. Introduction

The Nambu-Jona-Lasinio (NJL) model [1], which consists only of fermions, is still important as a model of dynamical symmetry breaking considered in theories beyond the standard model. Recently, Bunk et al. have studied the NJL model in

[*]Corresponding Author

spacetime with an extra dimension [2]. As in a non-renormalizable model, it is worth studying the effect of two different scales, the cutoff scale and the compactification scale.

In the present paper, we introduce the four-dimensional models similarly as in the view point of dimensional deconstruction [3] and investigate the one-loop quantum effect in a theory with a finite spectrum of fermions. In dimensional deconstruction models [3], excitation modes exist, similarly in higher-dimensional theory, and it is known that similar features are found as in higher-dimensional theory in a certain case. In a model which consists of four dimensional fermions of a limited number as in the dimensional deconstruction, we wish to investigate how quantum effects bring about symmetry breaking.

Two situations can be considered about chiral symmetry in construction of our models. One is the case where it has a mass spectrum based on a deconstruction model. In this case, quantum correction to the fermion mass would be observed and a large amount of condensation of fermion is expected for a certain strong coupling. Another situation is the case where chiral symmetry is imposed about all the fermion fields in the model. In this case, the condensation of fermions through quantum effect at strong coupling produces massive states of fermions. Even in this case, as seen later, the technique used in a deconstruction model is valid for obtaining the one-loop effect by a prescription using auxiliary fields.

The plan of this paper is as follows. In the next section, we consider Model A, in which the mass matrix of fermions is introduced. Similarity to a five dimensional four-fermion model will be discussed. In Section 3, we consider a massless NJL model with 'non-diagonal' four-fermion interactions, as Model B. The interacting term is chosen as the effective fermion masses have a spectrum akin to that in a deconstruction model. We end with a summary in Section 4.

2. Model A

In this section, we consider a type of a massive NJL model. The free part of the Lagrangian for fermion fields takes the form

$$\mathcal{L}_f = \sum_{k=1}^{N} [i\bar{\lambda}_k \partial \lambda_k + i\bar{\chi}_k \partial \chi_k] + \sum_{k=1}^{N} [\bar{\lambda}_k (m_0 \chi_k - f \chi_{k+1}) + \text{h.c.}], \qquad (2.1)$$

where λ_k and χ_k are Weyl spinors with opposite chiralities. The 'bare' mass m_0

and the constant f have dimension of mass and take real values. The $N+1$-st fields should be identified to the first one, i.e., $\chi_{N+1} \equiv \chi_1$. The eigenvalues for the fermion mass-squared are given by

$$\overline{M}_p^2 = m_0^2 + f^2 - 2fm_0 \cos \frac{2\pi p}{N}, \tag{2.2}$$

with $p = 0.1, ..., N-1$ and if and only if $m_0 = f$, a zero eigenvalue exists [5]. The eigenmodes turn out to be

$$\chi^{(p)} = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 \\ e^{i\frac{2\pi p}{N}} \\ e^{i\frac{4\pi p}{N}} \\ \vdots \\ e^{i\frac{2\pi(N-1)p}{N}} \end{pmatrix}. \tag{2.3}$$

Now, we introduce an interacting part of the Lagrangian. We assume it as

$$\mathcal{L}_G = G \sum_{k=1}^{N} (\overline{\lambda}_k \chi_k)(\overline{\chi}_k \lambda_k), \tag{2.4}$$

where G is the four-fermion coupling and the total Lagrangian is supposed to be $\mathcal{L} = \mathcal{L}_f + \mathcal{L}_G$. The four-fermi interaction is 'local' with respect to each entry of the fermion, and it is different from the interaction in the Gross-Neveu model [6]. Apparently, if $m_0 = f$, only the zero-mode fermion possesses chiral symmetry.

The interaction term can be replaced by the term including an auxiliary field. That is, the interacting part of the Lagrangian can be rewritten to

$$\mathcal{L}'_G = -m_0 \sum_{k=1}^{N} (\overline{\lambda}_k \chi_k + \overline{\chi}_k \lambda_k) + \sum_{k=1}^{N} [(\overline{\lambda}_k \chi_k)\sigma_k + (\overline{\chi}_k \lambda_k)\sigma_k^*]$$

$$- \sum_{k=1}^{N} \frac{|\sigma_k - m_0|^2}{G}, \tag{2.5}$$

where the complex fields σ_k $(k = 1, 2, ..., N)$ are auxiliary fields.

Throughout the present paper, we use a homogeneity ansatz

$$\sigma_1 = \sigma_2 = \cdots = \sigma_N \equiv \sigma, \tag{2.6}$$

and the modulation with respect to k is assumed to be negligible. Now, we can obtain the effective potential for σ up to one-loop order in the following form:

$$V(\sigma) = \frac{N|\sigma - m_0|^2}{G} + i\sum_{p=0}^{N-1} \text{Tr} \ln(i\partial + M_p), \tag{2.7}$$

where the effective mass of fermions are given by

$$M_p^2 = |\sigma - fe^{i\frac{2\pi p}{N}}|^2 = |\sigma|^2 + f^2 - 2f|\sigma|\cos\vartheta_p, \tag{2.8}$$

where $\vartheta_p = \dfrac{2\pi p}{N} - \arg(\sigma)$. The amount of fermion condensation is determined by the gap equation $\dfrac{dV}{d\sigma}(\langle\sigma\rangle) = 0$.

The one-loop effective potential is evaluated formally by Schwinger's parameter integral as

$$2\int \frac{d^4k}{(2\pi)^4} \int_0^\infty \frac{dt}{t} \sum_{p=0}^{N-1} \exp[-(k^2 + M_p^2)t]$$

$$= \frac{2}{(4\pi)^2} \int_0^\infty \frac{dt}{t^3} \sum_{p=0}^{N-1} \exp[-M_p^2 t]. \tag{2.9}$$

Furthermore, we can apply the formula [7]

$$e^{(2f^2\cos\vartheta)t} = \sum_{\ell=-\infty}^\infty e^{i\ell\vartheta} I_\ell(2f^2 t), \tag{2.10}$$

where $I_\ell(z)$ is the modified Bessel function, to the expression. Then we find

$$V(\sigma) = \frac{N|\sigma - m_0|^2}{G} + \frac{2}{(4\pi)^2} \sum_{p=0}^{N-1} \sum_{\ell=-\infty}^\infty e^{i\ell\vartheta_p} \mathcal{I}(\ell; f, |\sigma|), \tag{2.11}$$

where

$$\mathcal{I}(\ell;\, f,\,|\sigma|) \equiv \int_0^\infty \frac{dt}{t^3} \exp[-(f^2+|\sigma|^2)t]I_\ell(2f|\sigma|t). \qquad (2.12)$$

Now, the summation over p can be performed, and it turns out to be

$$V(\sigma) = \frac{N|\sigma - m_0|^2}{G} + NV_1(\sigma), \qquad (2.13)$$

with

$$V_1(\sigma) = \frac{2}{(4\pi)^2} \sum_{q=-\infty}^{\infty} e^{-iqN\,\arg(\sigma)}\mathcal{I}(qN;\, f,\,|\sigma|)$$

$$= \frac{2}{(4\pi)^2}\mathcal{I}(0;\, f,\,|\sigma|) + \frac{4}{(4\pi)^2}\sum_{q=1}^{\infty} \cos[qN\,\arg(\sigma)]\mathcal{I}(qN;\, f,\,|\sigma|). \quad (2.14)$$

The integral $\mathcal{I}(0;\, f,\,|\sigma|)$ is divergent as in the present form.[1] Therefore, some regularization is needed. Several regularization methods on Schwinger's parameter integral have been considered by many authors. A 'minimal subtraction' method [8] uses the cutoff method only for the divergent part as in the original NJL model. A use of density with cutoff [9] is also considered. In the present paper, we adopt the simple introduction of the cutoff scale Λ, as for the lower bound of t as Λ^{-2} [10-13]. Namely, we use $\mathcal{I}_\Lambda(\ell;\, f,\,|\sigma|)$ defined by

$$\mathcal{I}_\Lambda(\ell;\, f,\,|\sigma|) \equiv \int_{\Lambda^{-2}}^\infty \frac{dt}{t^3} \exp[-(f^2+|\sigma|^2)t]I_\ell(2f|\sigma|t), \qquad (2.15)$$

instead of $\mathcal{I}(\ell;\, f,\,|\sigma|)$.

It is obvious that the one-loop part of the potential, $V_1(\sigma)$, has a Z_N symmetry; $V_1(\sigma)$ is invariant under the change that $\arg(\sigma) \rightarrow \arg(\sigma) + \frac{2\pi}{N}$. A typical shape is shown in Figure 1 as in a density plot, where the brighter region means the larger value of the potential.

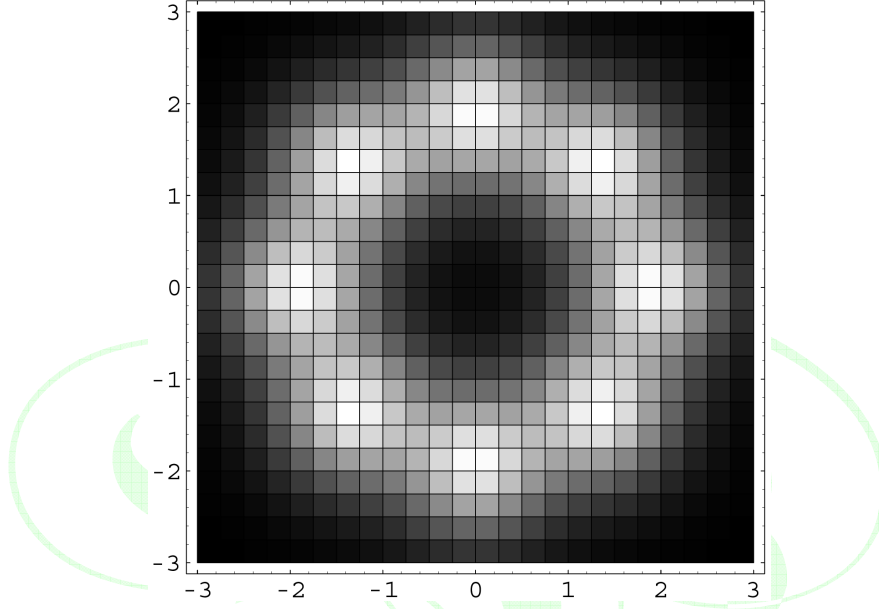[1]Note that $\mathcal{I}(\ell;\, f,\,|\sigma|)$ for $\ell \geq 3$ is finite.

Figure 1. A density plot of $V_1(\sigma)$ for $N = 8$ and $f/\Lambda = 2$. The axes indicate the real and imaginary part of σ.

We find that the maximum of V_1 is located at the value for $|\sigma|$ which is smaller than f. Thus, if the bare mass $m_0 = f$, the 'dynamical' mass $m = \langle\!\langle |\sigma| \rangle\!\rangle$ becomes larger than f, unless $G = 0$. This is due to the interaction term which mixes the eigenmodes of the free theory.

Conversely, the choice of m_0 can lead to the unmodified mass, $m = m_0$, for any small value for G. This can be achieved by setting the maximum of V_1 is located at $\sigma = m_0$. Then the condensation of fermions vanishes. In this case, let us denote the special value for mass as m_*, which is expressed as a function of f/Λ. In Figure 2, m_*/f is plotted against f/Λ. For large f/Λ, the value of m_* approaches to the value f.

Incidentally, we recognize an existence of the case that $m = f$ is possible for the bare mass in the range $m_* < m_0 < f$ with an appropriate tuning of G. Thus, when f/Λ becomes large, $(m - m_0)/f$ converges to zero also in this case. The qualitative features stated so far are independent of N.
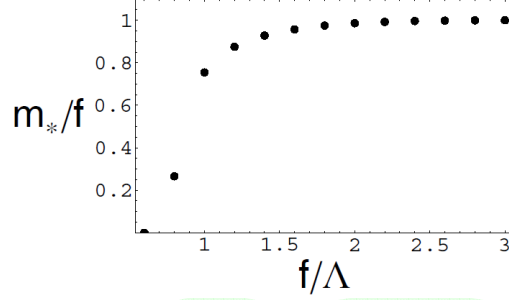
Figure 2. m_*/f is plotted against f/Λ for $N = 8$.

The critical coupling $G = G_c$ for fermion condensation is calculated from the second derivative $V(\sigma)$ at $m = m_*$. If $G > G_c$, the effective potential $V(\sigma)$ becomes a local maximum at $\sigma = m_*$. Then, condensation of fermions is expected to pick a large value and even the lowest mass eigenvalue of fermions acquires a large mass. We show the value of $G_c\Lambda^2$ for $m_0 = m_*$ in Figure 3, for relatively large values of f/Λ.
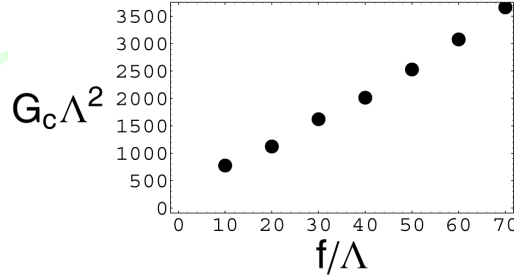

Figure 3. $G_c\Lambda^2$ in the case of $m_0 = m_*$ is plotted against f/Λ for $N = 8$.

Now, we further investigate the limiting case that f/Λ is large. For $N \gg 1$ and $f|\sigma|\Lambda^{-2} \gg 1$, we can estimate the one-loop effect as

$$V_1(\sigma) \approx \frac{2\Lambda^4}{(4\pi)^2} \int_1^\infty \frac{dt}{t^3} \exp[-(f^2 + |\sigma|^2)\Lambda^{-2}t]I_0(2f|\sigma|\Lambda^{-2}t)$$

$$\approx \frac{2\Lambda^4}{(4\pi)^2} \int_1^\infty \frac{dt}{t^3} \exp[-(f - |\sigma|)^2\Lambda^{-2}t]\frac{1}{\sqrt{4\pi f|\sigma|\Lambda^{-2}t}}. \qquad (2.16)$$

Therefore in the region $\sigma \approx f$, V_1 is approximated as

$$V_1(\sigma) \approx \frac{2\Lambda^5}{(4\pi)^{5/2} f} \int_1^\infty \frac{dt}{t^{7/2}} \exp[-(f - |\sigma|)^2 \Lambda^{-2} t]. \qquad (2.17)$$

The critical value for G_c calculated from the approximation appears to be proportional to f. This observation qualitatively agrees with the result shown in Figure 3.

The comparison to a higher-dimensional model is worth studying. Let us consider a five-dimensional interacting four-fermion model governed by the Lagrangian

$$\mathcal{L}_{KK} = i\overline{\psi}\partial_{(5)}\psi + G_5\overline{\psi}\psi\overline{\psi}\psi, \qquad (2.18)$$

where $\partial_{(5)}$ is the Dirac operator defined in five dimensions. By using a real auxiliary field S, we can rewrite this as[2]

$$\mathcal{L}_{KK} = i\overline{\psi}\partial_{(5)}\psi + S\overline{\psi}\psi - \frac{S^2}{4G_5}. \qquad (2.19)$$

Suppose that the fifth dimension is compactified to a circle whose circumference is L. Then, the one-loop effective potential for the auxiliary field S is given by

$$V_{KK}(\sigma) = \frac{S^2}{4G_5} + i\frac{1}{L} \sum_{p=-\infty}^{\infty} \mathrm{Tr}\ln(i\partial + M_p), \qquad (2.20)$$

where $M_p = S + i\gamma_5 \dfrac{2\pi p}{L}$.

For large value of $L\Lambda$, this expression can be approximated, using an identity for the elliptic theta function, as

$$V_{KK}(\sigma) \approx \frac{S^2}{4G_5} + \frac{2\Lambda^5}{(4\pi)^{5/2}} \int_1^\infty \frac{dt}{t^{7/2}} \exp\left[-\frac{S^2}{\Lambda^2} t\right]. \qquad (2.21)$$

---

[2]As is well known, the quality of approximation is guaranteed for large N fermion models. We consider here a simple model for explanation.

By comparison of two models, the critical coupling G_{5c} for condensation of the fermion in the five dimensional model is found to be related to the critical coupling G_c in the four dimensional model as the following manner:

$$G_{5c} \approx \frac{G_c}{4f}, \tag{2.22}$$

for large f/Λ. Because the relation between deconstruction models and Kaluza-Klein models is $L = N/f$ for large f and large N [3-5], the relation reads $G_{c5}/L \approx G_c/N$. The appearance of the characteristic number of the model, N, indicates that the five-dimensional model and the four-dimensional model considered here have a common feature but are not entirely equivalent in the limit. The dimensional deconstruction has a continuum limit to the higher-dimensional theory for free fields, but exact discretization is not realized by the method generally.

The difference in the degree of freedom of auxiliary fields between four and five dimensional model is due to the absence of chirality in five dimensions. We think that more similar five-dimensional model is the model with a $U(1)$ gauge field. The vector field in the compactified fifth dimension plays a role of the imaginary part of the order parameter. The analysis on such a model will be presented elsewhere.

3. Model B

The model B proposed in this section does not include mass scales as input parameters. This model mimics the model A, in view of the coupling between adjacent fermions. Simply speaking, the mass scale f is supposed to be replaced by an auxiliary field ρ_k. The Lagrangian of the model B consists of the free part (2.1) with $m_0 = f = 0$ and the interacting part, expressed by use of auxiliary fields,

$$\mathcal{L}'_G = \sum_{k=1}^{N} [(\bar{\lambda}_k \chi_k)\sigma_k + \text{h.c.}] - \sum_{k=1}^{N} [(\bar{\lambda}_k \chi_{k+1})\rho_k + \text{h.c.}]$$

$$- \sum_{k=1}^{N} \frac{\cosh\theta |\sigma_k|^2 - 2\sinh\theta \, \text{Re}(\sigma_k \rho_k^*) + \cosh\theta |\rho_k|^2}{G}, \tag{3.1}$$

where G and θ are constant. If the auxiliary fields are integrated out, the following

four-fermion interaction can be found:

$$\mathcal{L}_G = G \sum_{k=1}^{N} [\cosh \theta (\bar{\lambda}_k \chi_k)(\bar{\chi}_k \lambda_k) - \sinh \theta [(\bar{\lambda}_k \chi_{k+1})(\bar{\chi}_k \lambda_k) + (\bar{\lambda}_k \chi_k)(\bar{\chi}_{k+1} \lambda_k)]$$

$$+ \cosh \theta (\bar{\lambda}_k \chi_{k+1})(\bar{\chi}_{k+1} \lambda_k)]. \tag{3.2}$$

As in the previous section, we require a homogeneity ansatz,

$$\sigma_1 = \sigma_2 = \cdots = \sigma_N \equiv \sigma \text{ and } \rho_1 = \rho_2 = \cdots = \rho_N \equiv \rho. \tag{3.3}$$

Then, the fermion mass-squared spectrum is given by

$$M_p^2 = |\sigma - \rho e^{i\frac{2\pi p}{N}}|^2 = |\sigma|^2 + |\rho|^2 - 2|\sigma||\rho|\cos \vartheta_p, \tag{3.4}$$

where $\vartheta_p = \dfrac{2\pi p}{N} + \arg(\rho) - \arg(\sigma) \equiv \dfrac{2\pi p}{N} + \varphi$ and $p = 0, 1, ..., N - 1.$[3]

The effective potential in this model is expressed as

$$V(|\sigma|, |\rho|, \varphi)$$

$$= N \frac{\cosh \theta |\sigma|^2 - 2 \sinh \theta \cos \varphi |\sigma||\rho| + \cosh \theta |\rho|^2}{G} + N V_1(|\sigma|, |\rho|, \varphi) \tag{3.5}$$

with

$$V_1(|\sigma|, |\rho|, \varphi) = \frac{2}{(4\pi)^2} \mathcal{I}(0; |\rho|, |\sigma|) + \frac{4}{(4\pi)^2} \sum_{q=1}^{\infty} \cos[qN\varphi] \mathcal{I}(qN; |\rho|, |\sigma|). \tag{3.6}$$

In this paper, we assume $\theta > 0$ and $\langle \varphi \rangle = 0$ (unbroken CP). In Figure 4, we show the one-loop contribution $V_1(|\sigma|, |\rho|, 0)$ as a density plot for $N = 8$. In Figure 5, we show the effective potential $V(|\sigma|, |\rho|, 0)$ for the case with $G\Lambda^2 = 1000$, $\theta = 1$ and $N = 8$. We find that the origin is a local maximum and the minimum point exists evidently in this case. Thus chiral symmetry is broken, because $\langle |\sigma| \rangle = \langle |\rho| \rangle \neq 0$ and condensation of fermions occurs. Then, the effective fermion

---

[3]Thus, we know that $\arg(\rho) + \arg(\sigma)$ is the Nambu-Goldstone mode.

mass spectrum is dynamically generated as

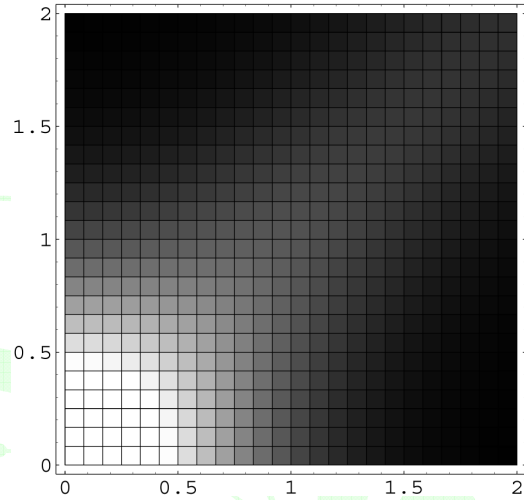$$M_p^2 = 4\langle |\sigma| \rangle^2 \sin^2 \frac{\pi p}{N}. \tag{3.7}$$

Figure 4. $V_1(|\sigma|, |\rho|, 0)$ is shown as a density plot for $N = 8$. The axes indicate $|\sigma|$ and $|\rho|$.
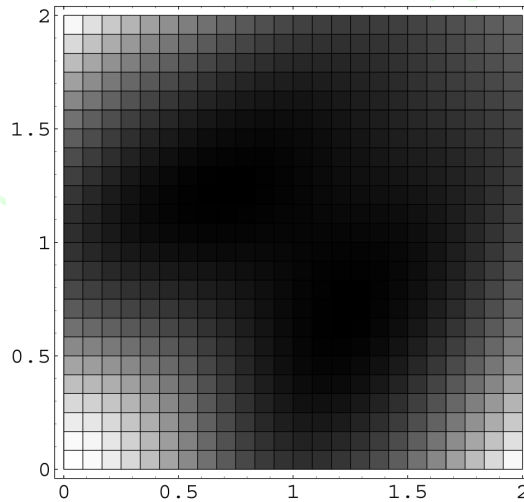

Figure 5. $V(|\sigma|, |\rho|, 0)$ is shown as a density plot for $G\Lambda^2 = 1000$, $\theta = 1$ and $N = 8$. The axes indicate $|\sigma|$ and $|\rho|$.

In this model, the critical coupling for condensation can be analytically evaluated as

$$G_c e^\theta \Lambda^2 = 8\pi^2,$$ (3.8)

and is found to be independent of N.

4. Summary and Outlook

In the present paper, we have considered two unorthodox extensions of the Abelian NJL model. The model A turns out to be related to a higher-dimensional model for a large mass scale $f/\Lambda \gg 1$. We find that the model B can exhibit a fermion mass spectrum as in the deconstruction models.

Because of the variety of the effective fermion mass, the effective potential at finite temperature or density may lead to various phases for condensation. The thermodynamic properties of the models should be studied also in cosmological models.

The models in this paper include self-interactions and interactions between adjacent fermion fields. We are interested in the general structure in the interactions, which may be deeply investigated with the knowledge of spectral graph theory.

We hope that applications of our toy models to phenomenological models will be explored in future.

References

[1]   Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122 (1961), 345.

[2]   D. Bunk, J. Hubisz, J. Shao and P. Tanedo, arXiv:1111.3951 [hep-ph].

[3]   N. Arkani-Hamed, A. G. Cohen and H. Georgi, Phys. Rev. Lett. 86 (2001), 4757.

[4]   C. T. Hill, S. Pokorski and J. Wang, Phys. Rev. D 64 (2001), 105005.

[5]   N. Kan and K. Shiraishi, J. Math. Phys. 46 (2005), 112301.

[6]   D. J. Gross and A. Neveu, Phys. Rev. D 10 (1974), 3235.

[7]   N. Kan, K. Sakamoto and K. Shiraishi, Euro. Phys. J. C 28 (2003), 425.

[8]   E. Elizalde, S. P. Gavrilov, S. D. Odintsov and Yu. I. Shil'nov, Braz. J. Phys. 30 (2000), 573.

[9]   B. Hiller, A. A. Osipov, A. H. Blin and J. da Providência, SIGMA 4 (2008), 024.

[10]   E. Elizalde, S. Leseduarte, S. D. Odintsov and Yu. I. Shil'nov, Phys. Rev. D 53 (1996), 1917.

[11]   I. L. Buchbinder, T. Inagaki and S. D. Odintsov, Mod. Phys. Lett. A 12 (1997), 2271.

[12]   M. Ishi-i, T. Kashiwa and N. Tanimura, Prog. Theor. Phys. 100 (1998), 353.

[13]   T. Inagaki, S. D. Odintsov and Yu. I. Shil'nov, Int. J. Mod. Phys. A 14 (1999), 481.